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Conceptual Knowledge of Decimal Arithmetic

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Abstract

In two studies (N's = 55 and 54), we examined a basic form of conceptual understanding of rational number arithmetic, the direction of effect of decimal arithmetic operations, at a level of detail useful for informing instruction. Middle school students were presented tasks examining knowledge of the direction of effects (e.g., "True or false: 0.77 * 0.63 > 0.77), number line estimation of decimal magnitudes, and knowledge of decimal arithmetic procedures. Their confidence in their direction of effect judgments was also assessed. We found (1) most students incorrectly predicted the direction of effect of multiplication and division with decimals below one; (2) this pattern held for students who accurately estimated the magnitudes of individual decimals and correctly executed decimal arithmetic operations; (3) explanations of direction of effect judgments that cited both the arithmetic operation and the numbers' magnitudes were strongly associated with accurate judgments; and (4) judgments were more accurate when multiplication problems involved a whole number and a decimal below one than with two decimals below one. Implications of the findings for instruction are discussed.

Keywords: *Rational Number Arithmetic, Decimal, Conceptual Knowledge, Mathematical Development, Mathematical Cognition*

Educational Impact And Implications Statement

Fraction and decimal arithmetic are crucial for later mathematics achievement and for ability to succeed in many professions, but acquiring these capabilities poses large difficulties for many students. The present study reveals a particularly striking type of misunderstanding that is likely to impede student's efforts to learn decimal arithmetic: poor knowledge of the direction of effect produced by multiplication and division of decimals between 0 and 1. In the study, most middle school students erroneously believed that multiplying two positive decimals below one must yield an answer greater than either of them (e.g., $0.77 * 0.63 > 0.77$), and that dividing by a decimal below one must yield an answer less than the number being divided (e.g., $0.77 \div 0.63 < 0.77$). The present study also demonstrated that students' judgments were more accurate when the multiplication problems included a whole number and a decimal (e.g., $5 * 0.291 < 5$) than when they included two decimals between 0 and 1, which suggested means for interventions to improve student's understanding of decimal arithmetic.

Conceptual Knowledge of Decimal Arithmetic

Understanding rational number arithmetic is central to a broad range of mathematical and scientific areas: algebra, geometry, trigonometry, statistics, physics, chemistry, biology, economics, and psychology, among them. One sign of this importance is that rational number arithmetic was part of more than half of the equations on the reference sheets for the most recent U.S. advanced placement physics and chemistry exams (College Board, 2014, 2015). Converging evidence comes from a longitudinal study of children's mathematics learning: In both the U.K. and the U.S., fifth graders' fraction and decimal arithmetic performance predicted their algebra knowledge and overall mathematics achievement in tenth grade, even after IQ, SES, race, ethnicity, whole number knowledge, reading comprehension, working memory, and other relevant variables were statistically controlled (Siegler et al., 2012). Beyond the classroom, rational number arithmetic is crucial for success not only in STEM areas but also in many occupations that do not require advanced math, including nursing, carpentry, and auto mechanic positions (e.g., Hoyles, Noss & Pozzi, 2001; Sformo, 2008). This importance of rational number arithmetic both inside and outside the classroom is one reason why the Common Core State Standards Initiative (CCSSI, 2015) recommended that a substantial part of math instruction in 3rd through 7th grades be devoted to the subject.

Despite years of classroom instruction, many students fail to master arithmetic with decimals and fractions (Bailey, Hoard, Nugent, & Geary, 2012; Booth, Newton, & Twiss-Garrity, 2014; Byrnes & Wasik, 1991; Hecht, 1998; Hecht & Vagi, 2010; Hiebert & Wearne, 1985; Mazzocco & Devlin, 2008; Siegler, Thompson, & Schneider, 2011). Consider a few representative examples: 1) U.S 8th graders who were tested on the four basic fraction arithmetic operations correctly answered only 57% of problems (Siegler & Pyke, 2013). 2) In a study of

U.S. 9^{th} graders, only 66% correctly answered the problem 4 + 0.3, only 65% correctly answered 0.05 * 0.4, and only 46% correctly answered 3 ÷ 0.6 (Hiebert & Wearne, 1985). 3) On a standardized test with a nationally representative sample (the NAEP: *National Assessment of Educational Progress*) presented in 1978 and in a controlled experiment with the same item in 2014, fewer than 27% of U.S. 8^{th} graders estimated correctly whether the closest answer to 12/13 + 7/8 was 1, 2, 19, or 21 (Carpenter, et al., 1980; Lortie-Forgues, Tian, & Siegler, 2015); 4) On the same NAEP, only 28% of U.S. 8^{th} graders correctly chose whether the closest product to 3.04 * 5.3, was 1.6, 16, 160, or 1600 (Carpenter et al.,1983).

The particular erroneous strategies that are used to solve rational number arithmetic problems convey the nature of the problem. With decimals, children often overgeneralize to multiplication the addition rule for placing the decimal point. They correctly answer that 1.23 + 4.56 = 5.79, but incorrectly claim that 1.23 * 4.56 = 560.88 (Hiebert & Wearne, 1985). Elementary, middle, and high school students also encounter difficulties when decimals involve one or more “0’s” immediately to the right of the decimal point; many ignore those 0’s and claim, for example, that 0.02 * 0.03 =0.6 (Hiebert & Wearne, 1986). Similar erroneous strategies often appear with common fractions (i.e., numbers expressed as N/M), for example treating numerators and denominators as independent whole numbers and operating on them separately (e.g., 1/2 + 1/2 = 2/4; Ni & Zhou, 2005).

These and related data have led numerous investigators to suggest that students lack conceptual understanding of rational number arithmetic. Within this view, which we share, lack of understanding of rational number arithmetic limits students’ ability to learn and remember the relevant procedures. For example, such lack of understanding could prevent students from rejecting implausible answers and the procedures that generated the answers and therefore lead the students not to search for more reasonable procedures.

Although the general conclusion is widely accepted, the specifics of what students do and do not understand about rational number arithmetic are largely unknown. Without these specifics, claims that students lack conceptual understanding have limited scientific use and few instructional implications. Therefore, the main purpose of the present study is to determine what middle school students do and do not understand about rational number arithmetic procedures, with an eye toward specifying the difficulties at a level useful for improving instruction.

In Study 1, we examined whether a particularly striking type of misunderstanding – direction of effect errors – are seen with decimals, as they previously have been documented to be with common fractions. We also examined children's confidence ratings of their direction of effect judgments to distinguish among several theoretical interpretations of the judgments. In Study 2, we determined whether direction of effect misconceptions extend to problems involving a whole number and a decimal and also obtained explanations of direction of effect judgments to better understand the reasoning underlying children’s judgments.

Direction of Effect of Rational Number Arithmetic Operations

Perhaps the most basic understanding about rational number arithmetic is the direction of effect that the operations produce: Will the answer be larger or smaller than the *operands* (the numbers in the problem). To examine knowledge of this type, Siegler and Lortie-Forgues (2015) devised a direction of effect task that presented inequalities such as the following: "True or False: $31/56 \times 17/42 > 31/56$". Fractions with relatively large numerators and denominators were used to prevent participants from calculating exact answers and thus answering correctly without considering the direction of effect of the arithmetic operation with those numbers.

For addition and subtraction of positive numbers, the direction is the same regardless of the size of the numbers: addition of positive numbers always yields an answer greater than either operand, and subtraction always yield an answer smaller than the number from which another number is being subtracted. However, for multiplication and division, the direction of effect varies with the size of the operands. Multiplying numbers above one always yields a product greater than either multiplicand, but multiplying numbers between zero and one never does. Conversely, dividing by numbers above one always results in answers less than the number being divided, but dividing by numbers between zero and one never does. Without understanding these relations, people cannot evaluate an answer's plausibility.

The implausible answers to rational number arithmetic problems that many students generate might be taken as evidence that students lack direction of effect knowledge. However, such answers might reflect students focusing on executing the computations and not considering the answer's plausibility. Rational number arithmetic imposes a high working memory load (English & Halford, 1995), which could lead to students not considering answers' plausibility. Therefore, to examine whether people reveal understanding of the direction of effect of fraction arithmetic when freed from the processing load imposed by computing, Siegler and Lortie-Forgues (2015) presented addition, subtraction, multiplication, and division direction of effect problems with fractions above one and fractions below one to sixth and eighth graders (12- and 14-year-olds) and pre-service teachers attending a highly ranked school of education.

The most striking finding of the study was that sixth graders, eighth graders, and pre-service teachers all were below chance in judging the direction of effect of multiplying and dividing fractions below one. For example, pre-service teachers erred on 67% of trials, and middle school students on 69% when asked to predict whether multiplying two fractions below one would produce an answer larger than the larger operand. These findings did not reflect weak knowledge of the fraction arithmetic procedures. The pattern was present even among the many pre-service teachers and children whose fraction arithmetic computation was perfect for the same operation, indicating that the inaccurate direction of effect judgments were not attributable to the teachers and students not knowing the relevant arithmetic operations. This observation attests to people being able to memorize mathematical procedures without even the most basic understanding of them. The findings also did not mean that the task was confusing or impossible. Math and science majors at a selective university erred on only 2% of the same problems.

These findings were not idiosyncratic to the task or samples. Highly similar findings emerged on a related item from the 2011 TIMSS (Trends in International Mathematics and Science Study), a standardized international comparison of math knowledge (Mullis, Martin, Foy, & Arora, 2011). Eighth graders were asked to judge which of four locations on a number line included the product of two unspecified fractions below one. The locations were: (a) between zero and the smaller multiplicand, (b) between the two multiplicands, (c) between the larger multiplicand and one, and (d) halfway between one and two. Consistent with Siegler and Lortie-Forgues' (2015) findings, 77% of U.S. 8^{th} graders erred on the problem.

These findings from both the experimental study and the large-sample international assessment raise the issue of whether difficulties understanding direction of effect of rational number arithmetic procedures are limited to fraction arithmetic or whether they reflect a more general difficulty in understanding multiplication and division of rational numbers, one that extends to decimals as well as fractions. It was entirely plausible that the difficulty with direction of effect judgments was limited to fractions. Fraction notation seems likely to 1) make it difficult to accurately estimate the magnitudes of individual numbers, which 2) increases the difficulty of estimating answers to arithmetic problems using those numbers, which 3) makes it difficult to recognize the implausibility of many answers yielded by incorrect fraction arithmetic procedures, which 4) makes it difficult to rule out these incorrect procedures, thereby reducing searches for correct procedures.

Consistent with the idea that fraction notation makes estimation of individual number's magnitude difficult, eighth graders' estimates for fractions between 0 and 5 are less accurate than second graders' estimates for whole numbers between 0 and 100 (Laski & Siegler, 2007; Siegler, Thompson, & Schneider, 2011). The greater difficulty of accurately estimating fraction magnitudes is unsurprising, because a fraction's magnitude must be derived from the ratio of the numerator and denominator rather than from a single number, as with whole numbers and decimals. Consistent with the idea that the fraction notation increases the difficulty of estimating answers to fraction arithmetic problems, middle school students are very inaccurate in estimating the answers to fraction arithmetic problems (Hecht & Vagi, 2010). Finally, consistent with the ideas that fraction notation makes it difficult to recognize implausible answer and rule out the wrong procedures that generated them, children frequently generate implausible fraction arithmetic answers, both through treating numerators and denominators as independent whole numbers ($1/2 + 1/2 = 2/4$) and through only operating on the numerator ($12/13 + 7/8 \cong 19$) (Lortie-Forgues, et al., 2015; Ni & Zhou, 2005). Thus, inaccuracy on the direction of effect task with fraction multiplication and division in Siegler and Lortie-Forgues (2015) and on the related TIMSS item might have reflected difficulties specific to fractions, especially difficulty accessing fraction magnitudes.

Another possibility, however, is that the inaccurate direction of effect judgments with fraction multiplication and division might reflect poor understanding of multiplication and division that extends beyond fractions and that has nothing directly to do with lack of magnitude understanding of individual numbers. In particular, participants might have overgeneralized the pattern of answers from whole number arithmetic and not understood that there is nothing about multiplication that requires answers to be greater than either operand and nothing about division that requires answers to be less than the number being divided. This interpretation suggests that weak understanding of multiplication and division should be as evident with decimals as with the corresponding fractions.

Supporting this latter interpretation, overgeneralizations from whole to rational numbers are very common with decimals, common fractions, and negatives alike. When comparing the magnitude of individual decimals, children often think that, as with whole numbers, more numerals implies larger magnitudes (e.g., claiming that .35 > .9; Resnick et al., 1989; Resnick & Omanson, 1987). Similarly, many children err on fraction magnitude comparison problems by assuming that fractions with larger whole number values for numerators and denominators are larger than fractions with smaller ones (e.g, 11/21 > 3/5; Fazio, Bailey, Thompson & Siegler, 2014; Ni & Zhou, 2005). Overgeneralization of whole number knowledge is also common with negative numbers (e.g., -12 > -6; Ojose, 2015).

Examining direction of effect judgments for decimal arithmetic provided a means for contrasting these two explanations. Unlike fractions, decimals are expressed by a single number, a feature that facilitates access to decimal magnitudes. To appreciate the difference, contrast the difficulty of judging the relative sizes of 7/9 and 10/13 with the ease of judging the relative sizes of their decimal equivalents, 0.78 and 0.77. Empirical data support this analysis; magnitude comparisons of college students are much faster and more accurate with decimals than fractions (DeWolf, Grounds, Bassok, & Holyoak, 2014). The same pattern holds for number line estimation as for magnitude comparison, and for children as well as adults (Iuculano & Butterworth, 2011; Desmet, Gregoire, & Mussolin, 2010).

Thus, if the inaccurate direction of effect judgments with multiplication and division of fractions between zero and one was due to difficulty accessing fraction magnitudes, then presenting the same task with decimals should reduce or eliminate the difficulty. If magnitude knowledge influenced direction of effect judgments, we also would expect individual children's accuracy on measures of the two types of knowledge to correlate positively. On the other hand, if inaccurate direction of effect judgments reflected limited understanding of multiplication and division, the same pattern should be evident with decimals as with fractions.

Our prediction was that the same difficulties with judging direction of effect for multiplication and division of operands between 0 and 1 would be present with decimals as had been documented previously with fractions. One source of support for this prediction was that when fourth and fifth graders were asked for their reaction to being told that 15 * 0.6 = 9, many children expressed surprise, with 25% saying without prompting that they expected the answer to be larger than 9 (Graber & Tirosh, 1990). Similar reactions were observed in the same study when students were told that 12 ÷ 0.6 = 20. Another paradigm has yielded similar results: When presented operands and answers and asked to select the appropriate operation, both high school students and pre-service teachers generally chose multiplication when problems yielded answers larger than the numbers being multiplied, and they chose division when problems yielded answers smaller than the number being divided, regardless of the semantics of the problem (Fischbein, Deri, Nello, & Marino, 1985; Tirosh & Graeber, 1989). Moreover, in previous studies of decimal arithmetic, students have been found to often misplace the decimal point on multiplication and division problems in ways that reflected little understanding of the plausibility of the answer (Hiebert & Wearne, 1985; 1986).

However, there was reason to hope that these findings underestimated current students' conceptual understanding of decimal arithmetic. One consideration was that the prior findings with decimals are 25 or more years old; the increased educational emphasis on conceptual understanding of rational numbers in recent years (e.g., CCSSI, 2015) might have increased understanding of decimal arithmetic among contemporary students. Moreover, the prior findings might underestimate children's understanding of decimal arithmetic: the participants tested had either had very little experience with decimal arithmetic (Graber & Tirosh, 1990) or the questions consisted of word problems, which often require complex verbal processing in addition to mathematical understanding (Fischbein, Deri, Nello, & Marino, 1985; Tirosh & Graeber, 1989).

A second purpose of Study 1 was to examine students' confidence in their direction of effect judgments. On mathematics problems, people sometimes generate wrong answers that they believe are correct; at other times, they generate wrong answers that they doubt are correct but cannot generate more likely alternatives. Participants in Siegler & Lortie-Forgues (2015) might have been convinced that their incorrect direction of effect judgments were correct, but they might have been unsure and relied on their whole number knowledge as a default option because they did not know what else to do. This type of default explanation seems to be common when people have limited knowledge of a topic (see Rozenblit & Keil, 2002 for examples of default explanations in non-mathematical contexts).

Obtaining confidence ratings allowed us to distinguish among three theoretical interpretations of incorrect direction of effect judgments on multiplication and division with decimals below one: (1) The *strong conviction hypothesis*, which posits that students are highly confident that multiplication produces answers greater than either operand and division produces answers less than the dividend; (2) The *operation knowledge hypothesis*, according to which students recognize that they know less about multiplication and division than addition and subtraction, and therefore are less confident in their multiplication and division judgments, regardless of whether the operands are below or above one; 3) The *cognitive conflict hypothesis*, in which, due to the contradiction between children's whole number experience and their experience multiplying and dividing numbers between zero and one, they are less confident in their multiplication and division direction of effect judgments with numbers between zero and one than in their other judgments.

If the strong conviction hypothesis is correct, confidence ratings for all eight types of problems should be equally high. If the operation knowledge hypothesis is correct, confidence ratings for the four addition and subtraction problems should be higher than for the four multiplication and division problems. If the cognitive conflict interpretation is correct, confidence ratings for multiplication and division of decimals below one should be lower than for the other six types of problems. Combinations of these alternatives were also possible; for example, children might be less confident in their multiplication and division judgments on all problems, and especially unconfident of judgments when those operations involve operands between 0 and 1.

Study 1

Method

Participants. The children were 55 middle school students (19 6th and 36 8th graders; 27 boys, 28 girls, Mean age = 12.75 years, SD = 1.06) who attended a public school in a middle-income suburban area near Pittsburgh, PA, U.S. These age groups were chosen because decimals were taught in the children's schools in fifth and sixth grades prior to the study and because doing so allowed direct comparison between direction of effect knowledge for fractions, which was examined in Siegler and Lortie-Forgues (2015), and for decimals, which was examined here. The school district included 59% Caucasian, 35% African American, 1% Asian, and 5% "other" children. Math achievement test scores were average for the state; 76% of 6^{th} graders and 81% of 8^{th} graders were at or above grade level, versus 78% and 75% for the state. Students were tested in groups in their math classroom during a regular class period in the middle of their school year.

Tasks.

Direction of effect judgments and confidence ratings. This task included 16 mathematical inequalities, four for each arithmetic operation. Each item was of the form "True or false: $a * b > a$?" Both a and b were two-place decimals, and a was always larger than b. On half of the problems, both a and b were below one (e.g., 0.77 * 0.63 > 0.77); on the other half, both were above one (e.g., 1.36 * 1.07 > 1.36). The same pairs of operands -- 0.77 and 0.63, 0.94 and 0.81, 1.36 and 1.07, and 1.42 and 1.15 -- were presented with all four arithmetic operations. Four problems, one with each arithmetic operation, were presented on each page of a booklet that children received; each pair of operands was used once on each page. Students received one point for each correct judgment.

After each problem, children were asked to rate their confidence in their answer on a 5-point scale ranging from "not confident at all" (1) to "extremely confident" (5). The numerical value of each confidence rating constituted the data on that trial; effects of arithmetic operation and operand size (above or below one) on the confidence ratings were analyzed.

Arithmetic computation. Participants were asked to answer 12 computation problems, 3 for each arithmetic operation. For each arithmetic operation, the operand pairs were 0.9 and 0.4, 0.45 and 0.18, and 3.3 and 1.2. The task was included to examine whether computation skill was related to understanding of direction of effects of the arithmetic operations.

Magnitude comparison. Children were presented 32 problems requiring comparison of 0.533 to another decimal. Half of the decimals were larger and half smaller than 0.533; equal numbers of these comparison numbers had 1, 2, 3, or 4 digits to the right of the decimal.

Procedure.

Tasks were always presented in the order: direction of effect, arithmetic computation, and magnitude comparison. Items within each task were presented in one of two orders, either first to last or last to first. All tasks were presented in printed booklets, with students writing answers with pencils. Students were asked to perform the problems in order; use of calculators was not allowed. The experiment was conducted by two research assistants and the first author.

Reliabilities.

Reliabilities of the measures (Cronbach's alpha) were above the satisfactory value of 0.70 (Nunnally, 1978), except in cases where ceiling effects were present, a factor known to lower reliabilities (May, Perez-Johnson, Haimson, Sattar, & Gleason, 2009). One case where ceiling effects were present and appeared to lower reliability involved the internal consistency of direction of effect judgments. The relatively low coefficient alpha on this task, $\alpha = 0.68$, appeared to be due to a ceiling effect on problems where performance was highly accurate and therefore where there was little variability. These were problems involving all four arithmetic operations when operands were above one and addition and subtraction problems with operands below one. More than half of students (56%) were 100% accurate on these 12 problems. On direction of effect problems where performance varied to a greater extent (multiplication and division of numbers below one), internal consistency was adequate (α= 0.74 and 0.80, respectively). Low internal consistency on the arithmetic computation task, α= 0.58, also appeared due to ceiling effects. In this case 64% of students correctly answered all addition and subtraction computation problems. Again, internal consistency on multiplication and division computation problems, where performance was more variable, was adequate (α = 0.71 and 0.76, respectively). Reliability of confidence ratings for direction of effect judgments was high (α= 0.95), as was internal consistency of magnitude comparisons (α = 0.94). See Online Supplemental Table S1 for the results presented separately for each grade on each task.

Results and Discussion

Direction of effect judgments. A repeated-measures ANOVA with decimal size (above or below 1) and arithmetic operation (addition, subtraction, multiplication, or division) as within-subject factors, grade (6^{th} or 8^{th}) as a between-subject factor, and number of correct direction of effect judgments as the dependent variable yielded main effects of arithmetic operation ($F(3, 159) = 52.61$, $p < 0.001$, $\eta p2 = 0.49$) and decimal size ($F(1, 53) = 63.02$, $p < 0.001$, $\eta p2 = 0.54$), as well as a decimal size X arithmetic operation interaction ($F(3, 159) = 38.24$, $p < 0.001$, $\eta p2 = 0.42$). Post-hoc comparisons with the Bonferroni correction showed that number of correct predictions for decimals below and above one did not differ on addition (87% vs. 88% correct; $t(54) = 0.63$, $p = 0.53$) or subtraction (89% vs. 90%; $t(54) = 0.29$, $p = 0.77$), but differed greatly on multiplication (20% vs. 84%; $t(54) = 7.12$, $p < 0.001$) and division (19% vs. 89%; $t(54) = 9.04$, $p < 0.001$). Accuracy was below the chance level of 50% with decimals below one for both multiplication ($t(54) = 6.05$, $p < 0.001$) and division ($t(54) = 6.49$, $p < 0.001$).

Analysis of individual children's judgments showed similar findings. Half (49%) of students erred on all 4 multiplication and division problems with operands below one and correctly answered all 12 other problems.

As shown in Table 1, these direction of effect judgements with decimals mirrored previous data with fractions, with the single exception that decimal division judgments with operands below one were *less* accurate than the corresponding fraction judgments. The parallel patterns suggest that students' performance reflected a misunderstanding of multiplication and division that is independent of the numbers' format (see Online Supplemental Table S2 for the percentages for each grade reported separately).

- - Insert Table 1 about here - -

Confidence ratings. Confidence ratings for the direction of effect task were analyzed via a parallel repeated-measures ANOVA with decimal size and arithmetic operation as within-subject factors and grade as a between-subject factor. The analysis yielded a main effect of arithmetic operation ($F(3, 159) = 20.31$, $p < 0.001$, $\eta p2 = 0.28$), and a decimal size X grade interaction ($F(1, 53) = 4.63$, $p = 0.036$, $\eta p2 = 0.08$). Post-hoc comparisons with the Bonferroni correction showed that confidence in direction of effect judgments was lower for division (M = 3.97, SD = 1.01) than for multiplication (M = 4.37, SD = 0.72; $t(54) = 4.24$, $p < 0.001$), and lower for multiplication than for addition (M = 4.53, SD = 0.63; $t(54) = 2.61$, $p = 0.01$) or subtraction (M = 4.56, SD = 0.60; $t(54) = 3.17$, $p < 0.01$). The decimal size by grade interaction reflected 8th but not 6th graders being less confident in their judgments on problems with decimals below one than on problems with decimals above one (for 8th graders, mean confidence rating of 4.31 vs. 4.41, $t(35) = 2.64$, $p = 0.01$; for 6th graders, mean rating of 4.38 vs. 4.34, $t(18) = 0.77$, $p = 0.45$).

We next examined confidence ratings of the half of participants (49%) whose judgments always matched the direction of effect of arithmetic with whole numbers (i.e., always wrong on the 2 multiplication and 2 division problems with decimal operands below one and always correct on the other 12 problems). The analysis yielded a main effect of arithmetic operation ($F(3, 75) = 11.48$, $p < .001$, $\eta p2 = 0.315$). Confidence in direction of effect judgments was lower for division (M = 4.05, SD = 1.09) than for addition (M = 4.59, SD = 0.44; $t(26) = 3.29$, $p = 0.002$), subtraction (M = 4.58, SD = 0.48; $t(26) = 3.60$, $p < 0.001$), and multiplication (M = 4.58, SD = 0.53; $t(26) = 3.46$, $p = 0.002$). Confidence ratings did not differ between problems with numbers above and below one (for problems above one, M = 4.44, SD = 0.58; for problems below one, M = 4.46, SD = 0.58; $t(26) = 0.42$, $p = 0.7$).

In contrast, conducting the same analysis on the 51% of participants whose judgments did not invariably follow the direction of effect of whole number arithmetic yielded a decimal size X grade interaction ($F(1, 26) = 7.92$, $p = 0.009$, $\eta p2 = 0.233$) as well as a main effect of arithmetic operation ($F(3, 78) = 11.25$, $p < .001$, $\eta p2 = 0.302$). The main effect reflected lower confidence in division judgments (M = 3.89, SD = 0.95) than in ones for multiplication (M = 4.17, SD = 0.82; $t(27) = 2.51$, $p = 0.018$), and for multiplication judgments than for addition (M = 4.46, SD = 0.77; $t(27) = 3.21$, $p = 0.003$) and subtraction (M = 4.54, SD = 0.71; $t(27) = 4.47$, $p < 0.001$) ones. The interaction arose from 8th graders being less confident in their judgments on problems with decimals below than above one (M's = 4.09 and 4.27, SD's = .84 and .80; $t(18)$ 2.96, $p = 0.008$), but no difference being present for 6th graders (M’s = 4.50 vs. 4.38, $t(8)$ 1.37, $p = 0.206$). This interaction suggested that by 8th grade, children began to recognize that there was something different about computations with decimals below one than decimals above one.

Arithmetic computation. A repeated-measures ANOVA on accuracy of decimal arithmetic computation, with arithmetic operation as a within-subject factor, grade as a between-subject factor, and number of correct answers as the dependent variable yielded a main effect of arithmetic operation ($F(3, 159) = 51.5$, $p < 0.001$, $\eta p2 = 0.493$). Post-hoc comparisons with the Bonferroni correction showed that number correct was lower on division problems (M = 35%, SD = 40%) than on multiplication problems (M = 54%, SD = 39%) ($t(54) = 3.08$ $p < 0.01$) and lower on multiplication than on addition (M = 90%, SD = 18%) ($t(54) = 5.98$, $p < 0.001$) and subtraction problems (M = 93%, SD = 18%) ($t(54) = 6.23$, $p < 0.001$). There was no effect of grade, but 8^{th} graders tended to generate more correct answers on multiplication (6^{th} graders 44%; 8^{th} graders 59%) and division (6^{th} graders 21%; 8^{th} graders 43%) problems.

Decimal arithmetic accuracy (68% correct) closely resembled that on similar problems 30 years ago (e.g., Hiebert & Wearne, 1985). Also as then, misplacing the decimal point in the answer was the most common source of multiplication errors. On 73% of multiplication errors (34% of answers), students multiplied correctly but misplaced the decimal in the answer. Misplacing the decimal was also a fairly frequent source of division errors (21% of errors, 13% of answers).

The below chance direction of effect judgment accuracy on multiplication and division of decimals below one was not attributable to the less accurate computation on those operations. Most students (14 of 19, 74%) who correctly solved both multiplication computation problems involving decimals below one were incorrect on both of the direction of effect judgments on parallel problems. Similarly, among students who correctly answered both of the division computation problems with decimals below one, most (9 of 14, 64%) erred on both of the corresponding direction of effect problems.

For both 6th and 8th graders, numbers of correct arithmetic computations and direction of effect judgments were weakly correlated or uncorrelated (6th grade, r = -0.28, n.s.; 8th grade, r = 0.33, p = 0.05). The pattern was similar when the problems of greatest interest were analyzed separately. No relation was present when only multiplication direction of effect problems with operands below one and multiplication computation problems with operands below one were considered (6th grade, r = 0.26, n.s.; 8th grade, r = 0.13, n.s.) or when only division direction of effect problems with operands below one and division computation problems with operands below one were considered (6th grade, r = 0.37, n.s.; 8th grade, r = 0.19, n.s.).

Magnitude Comparison. Children correctly answered 83% of decimal magnitude comparisons. Performance was higher when the two decimals being compared had the same number of decimal places than when they had different numbers of decimal places (90% versus 80% correct, $t(54)$ 3.34, p = 0.002). Accuracy did not differ significantly between 6th and 8th graders, 77% versus 86%, $t(54) = 1.53$, p = 0.13.

Analyses of magnitude comparison errors showed large individual differences in knowledge of decimal magnitudes. At one extreme, 53% of children correctly answered more than 95% of decimal comparisons. At the other extreme, 18% of children answered incorrectly more than 90% of the 12 items on which ignoring the decimal point yielded a wrong answer (e.g., saying that 0.9 is smaller than 0.533, because 9 < 533).

For both 6th and 8th graders, numbers of correct magnitude comparison and direction of effect judgments were unrelated (6th grade, r = 0.01, n.s.; 8th grade, r = 0.10, n.s.). The same was true when only multiplication direction of effect problems with operands below one were considered (6th grade, r = 0.17, n.s.; 8th grade, r = -0.05, n.s.) and when only division direction of effect problems with operands below one were (6th grade, r = -0.03, n.s.; 8th grade, r = 0.10, n.s.).

In summary, direction of effect judgments with decimals were much like those observed by Siegler and Lortie-Forgues (2015) with common fractions. The 6th and 8th graders erred more often than chance on problems involving multiplication and division of decimals below 1, but were highly accurate on all other types of problems. These results with decimals could not be attributed to lack of magnitude knowledge. With both problems in general and on the two types of problems that elicited inaccurate direction of effect judgments, accuracy of magnitude comparison performance and direction of effect judgments were at most weakly related.

Study 2 was designed to build on these findings by examining direction of effect judgments on a type of problem that was potentially important for instruction -- problems that include a whole number and a decimal. Such problems provide a possible transition context through which instruction could build on students' understanding of whole number arithmetic and extend it to decimals. Study 2 also was designed to deepen our understanding of children's thinking about direction of effect judgments by having them explain their reasoning on them. As will be seen, the explanations proved invaluable for demonstrating that accurate predictions sometimes reflect processes quite different than the ones on which the predictions were based.

Study 2

In some U.S. textbooks series, such as *Everyday Math* (Bell et al., 2007) and *Prentice Hall Mathematics* (Charles et al., 2012), problems involving a whole number and a decimal below one are presented quite often. A likely reason is that such problems can capitalize on students' familiarity with whole numbers and with the usual framing of whole number multiplication as repeated addition. For instance, 5 * 0.34 can be interpreted as five iterations of 0.34. Even the phrasing "5 times 0.34" supports this interpretation. In contrast, the repeated addition interpretation is difficult to apply to multiplication if both operands are below one (viewing 0.05 * 0.3 as 0.3 being added 0.05 times is less intuitive than viewing 5 * 0.03 as 0.3 being added 5 times).

Because the repeated addition interpretation applies more straightforwardly to multiplication problems with a whole number and a decimal (*WD problems*) than to problems with two decimals (*DD problems*), direction of effect judgments for multiplication might be more accurate on WD than DD problems. Children could solve direction of effect problems with a whole and a decimal below one by estimating the result of adding the decimal the whole number of times; this logic is much more difficult to apply to problems with two decimals. However, students might not use the repeated addition interpretation of multiplication on either type of problem, because they were so convinced that multiplication always produces answers larger than the operands that they did not consider other possibilities, because they did not think of the repeated addition interpretation, or because they relied on some other interpretation. Thus, one goal of Experiment 2 was to test whether direction of effect judgments were more accurate on WD than DD multiplication problems.

At first glance, the same logic would seem to apply to division. For example, 3 ÷ 0.5 could be solved by six additions of 0.5, and children could solve the corresponding direction of effect problem by estimating the number of times 0.5 would need to be added to reach 3. However, several considerations suggested that for division, direction of effect problems would be no easier on WD than on DD problems. Although repeated addition and subtraction can be used to solve some WD division problems (ones where the dividend is bigger than the divisor and that have a whole number answer), the most common interpretation of division appears to be equal sharing (Carpenter, et al., 1999; Rizvi & Lawson, 2007). That interpretation makes sense with whole numbers (e.g., 30 ÷ 3 means 30 cookies shared equally among 3 friends), but is meaningless with decimal divisors (e.g., what does it mean to share 30 cookies among 0.3 friends). Because the equal sharing interpretation is not easily applicable to problems with decimal divisors, and because the repeated addition interpretation is useful for understanding only on a subset of division WD problems, we did not expect a difference between direction of effect judgment accuracy on WD and DD division problems.

A second main goal of Study 2 was to deepen our analysis of conceptual understanding of rational number arithmetic by asking students to explain the reasoning underlying their judgments on the direction of effect task. We were particularly interested in testing whether they apply the logic of repeated addition more often to WD than DD multiplication problems, and whether this logic underlay the predicted greater accuracy on WD than DD problems.

Method

Participants. Participants were 54 7th graders (26 boys, 28 girls, mean age = 12.7 years, SD = 0.54) who attended a public school in a middle-income suburban area near Pittsburgh, PA, U.S. The school district included 63% Caucasian, 22% African American, 7% Asian, 2% Hispanic, and 7% "other" children. As in Experiment 1, the school's mean math achievement was similar to that in the state as a whole (79% of 7^{th} graders in the district were at or above grade level, 73% in the state). Students were tested in groups in their math classroom during a regular class period near the end of the school year. A research assistant and a postdoctoral student (the first author) collected the data.

Tasks.

Direction of effect judgment only task. Each student was presented 36 problems (18 DD and 18 WD items). For each type of problem, there were six addition, six multiplication and six division items. Subtraction items were not presented in order to reduce the duration of the experiment and because direction of effect judgments on addition and subtraction problems were almost identical in the previous experiment.

Half of the DD problems for each operation involved pairs of decimals below one; the other half involved pairs of decimals above one. All WD problems for each operation included a whole number above one; half of these items included a decimal below one and half a decimal above one. On all WD problems, the whole number appeared first, the decimal appeared second, and the comparison answer was the whole number (e.g., “True or false: 5 * 0.291 > 5”). Problems were generated using one of the following sets of operand pairs:

Set A DD problems: 0.87 and 0.291; 0.96 and 0.173; 0.79 and 0.356; 8.83 and 3.584; 6.14 and 5.781; 12.87 and 2.854;

Set A WD problems: 5 and 0.291; 4 and 0.173; 14 and 0.356; 8 and 3.584; 6 and 5.781; 12 and 2.854.

Set B DD problems: 0.76 and 0.182; 0.85 and 0.261; 0.97 and 0.345; 9.74 and 5.495; 7.26 and 3.853; 11.49 and 2.898;

Set B WD problems: 6 and 0.182; 8 and 0.261; 13 and 0.345; 9 and 5.495; 7 and 3.853; 11 and 2.898)

DD problems were presented consecutively, as were WD problems. Problem order (DD problems first or WD problems first) and problem set (DD problems from set A and WD problems from set B, or vice versa) were counterbalanced. The items in Set A and Set B were chosen to be as similar as possible.

Judgment plus explanation task. The format of this task was identical to that of the judgment only task, except that students were asked to explain their reasoning immediately after each judgment. Such immediately retrospective strategy reports have been found to yield valid and non-reactive data for many numerical tasks, including arithmetic and number line estimation (e.g., Siegler, 1987; Siegler, et al., 2011). Presenting both the judgment only task and the judgment plus explanation task allowed us to obtain explanations data and also to test whether obtaining explanations affected judgments.

Each student was presented with 12 judgment plus-explanation problems (6 DD and 6 WD problems; two addition, two multiplication, and two division problems within each group; half with operands above one, and half with operands below one). Each problem was generated using one of two sets of operand pairs:

Set A DD items: 0.87 and 0.291; 8.83 and 3.584;

Set A WD items: 5 and 0.291; 8 and 3.584;

Set B DD Items: 0.76 and 0.182; 9.74 and 5.495;

Set B WD Items: 6 and 0.182; 9 and 5.495.

For each participant, order of problems (DD or WD first) was the same as on the judgment only task, but the sets of operand pairs used to generate the problems were switched. Participants whose DD problems on the judgment-only task were from Set A were presented DD problems on the judgment-plus-explanation task from Set B, and vice-versa.

Magnitude comparison. The task was the same as in Experiment 1, except that the problems where the decimals being compared had the same number of decimal places were excluded. This resulted in 24 decimal magnitude comparison problems.

Procedure.

The three tasks were presented in booklets in the order 1) direction of effect judgment-only task, 2) direction of effect judgment-plus-explanation task, 3) magnitude comparison task.

Children were asked to complete the tasks without a calculator in the order in which they appeared in the booklet.

Reliabilities of measures.

Measures of internal consistency (Cronbach's alpha) of the direction of effect judgment only task, the judgment plus explanation task, and the magnitude comparison task were all satisfactory ($\alpha = 0.74$, 0.71 and 0.95, respectively).

Results and Discussion

Direction of effect judgment-only task. We computed a repeated-measures ANOVA with decimal size (above or below one), arithmetic operation (addition, multiplication, or division) and whole number operand (present or absent) as within-subject factors; problem set (A or B) and problem order (DD first or WD first) as between-subject factors; and number of correct judgments as the dependent variable.

Main effects emerged for arithmetic operation ($F(2, 88) = 80.21$, $p < 0.001$, $\eta p2 = 0.646$) and decimal size ($F(1, 44) = 79.43$, $p < 0.001$, $\eta p2 = 0.644$). Three interactions also were present: arithmetic operation X whole number operand ($F(2, 88) = 3.49$, $p = 0.035$, $\eta p2 = 0.073$), arithmetic operation X decimal size ($F(2, 88) = 39.80$, $p < 0.001$, $\eta p2 = 0.475$), and arithmetic operation X whole number operand X decimal size ($F(2, 88) = 3.48$, $p = 0.035$, $\eta p2 = 0.073$).

The three-way interaction and the two two-way interactions could be interpreted quite straightforwardly. As shown in the three rows at the top of Table 2, when both operands were above one, answers were uniformly accurate on all three arithmetic operations. Neither arithmetic operation nor presence of a whole number affected accuracy on these problems. The high accuracy seems attributable to the direction of effect being the same for decimals as for whole numbers.

- - Insert Table 2 about here - -

As shown in the three rows at the bottom of Table 2, the pattern differed with decimals below one. On these problems, addition judgments were accurate and division problems inaccurate regardless of whether the problem included a whole number. These findings also appeared due to generalization from effects of the operation with whole numbers. In contrast, and consistent with our prediction, on multiplication problems with decimals below one, direction of effect judgments were more accurate when one operand was a whole number (WD problems) (M = 47%, SD = 47%) than when both operand were decimals (DD problems) (M = 31%, SD = 42%) ($t(53) = 2.97$, $p < 0.01$, Cohen's d = 0.41). This pattern was consistent across problems; direction of effect judgments were more accurate on all three multiplication problems that involved a whole number and a decimal below one (43%-50% correct) than on any of the three multiplication problems that involved two decimals below one (30%-33% correct). Consistent with this interpretation, accuracy with decimals below one was below the chance level (i.e., 50%) on multiplication DD problems ($t(53) = 3.35$, p < 0.001); division DD problems ($t(53) = 6.19$, p < 0.001); and division WD problems ($t(53) = 5.52$, p < 0.001); but not on multiplication WD problems ($t(53) = 0.48$, p = 0.63).

Analysis of individual children's direction of effect judgments yielded findings consistent with this interpretation. The number of students accurate on 100% of the WD problems was very similar to the number of participants accurate on 100% of the DD problems in every combination of arithmetic operation and decimal size, except for multiplication problems with decimals below one. On multiplication problems with decimals below one, almost twice as many children were correct on all three WD problems as on all three DD problems (39% vs. 22% of the sample).

Judgment-plus-explanation task. Comparing the leftmost two columns with the rightmost two columns of Table 2 indicated that judgment accuracy was very similar when explanations were and were not sought. Therefore, analyses of the judgment-plus-explanation task focus on the explanations. All explanations were classified independently by two raters. Percent agreement was 91% (Cohen's kappa was 0.85, above the adequate value of 0.75; Fleiss, 1981). Discussion between the raters was used to resolve discrepancies.

Most explanations (89%) fell into one of three categories:

1) *Operation-and-operand explanations* (14% of trials): Statements referring to both the operation and the operands or type of operands: "Multiplying with very small decimals makes the value of larger numbers go down"; "If you are multiplying by a number less than one, you will get a lower outcome".
2) *Unconditional operation explanations* (56%): Statements about an operation without reference to the operands or type of operand. This category includes rules such as: "Multiplication makes bigger" and "When you divide, the number decreases". Also included in this category are statements that implicitly assume that the effect of an operation is the same regardless of the type of operands (e.g., "9.74 * 5.495 will be greater than 9.74 because it's multiplication").
3) *Computational estimation explanations* (19%): Statements based on rounding of operands and approximate computation (e.g., for 9.74 * 5.495 > 9.74: "Greater because 9*5 is 45, which is greater than 9.74").

The remaining explanations were labeled "*Uninformative*" (11%). Of these, 8% could not be categorized (e.g., "because I know" or "you are making the number smaller"), and 3% where the child did not advance an explanation or the explanation was lost.

Frequency of each type of explanation varied with features of the problems. We examined these relations separately for each type of explanation.

Operation-and-operand explanations. Frequency of operation-and-operand explanations varied with the operation (χ^2 (2, 648) = 15.45, p < 0.01). It was less common on addition (6% of trials) than on multiplication (19%; χ^2 (1, 432) = 15.68, p < 0.01) and division (14%; χ^2 (1, 432) = 7.46, p < 0.01). The difference is consistent with the fact that operand size is irrelevant to the direction of effect for addition of positive numbers, but it does influence direction of effect for multiplication and division, making citation of operand size relevant for them.

Frequency of operation-and-operand explanations also varied with the size of the operands, but only on multiplication problems. Such explanations were more common on multiplication problems with decimals below than above one (25% versus 12%; χ^2 (1, 216) = 6.01, p = 0.01). Frequency of operation-and-operand explanations did not differ significantly between DD (10%) and WD (15%) problems.

Unconditional operation explanations. Frequency of unconditional operation explanations varied with the arithmetic operation (χ^2 (2, 648) = 15.45, p < 0.01). They were more common on addition (60% of trials) and division (61%) than on multiplication (50%).

Computational estimation explanations. Frequency of computational estimation explanations varied with the arithmetic operation (χ^2 (2, 648) = 20.74, p < 0.01). They were less frequent with division (10% of trials) than with multiplication (19% of trials; χ^2 (1, 432) = 6.71, p = 0.01) and addition (27% of trials; χ^2 (1, 432) = 20.80, p < 0.01). Lower frequency of computational estimation on division problems is consistent with it being less well understood than the other arithmetic operations (Carey, 2011; Foley & Cawley, 2003).

Computational estimation explanations were also more common on problems with decimals above one than below one, but only for multiplication (32% versus 7% of trials; χ^2 (1, 216) = 21.95, p = 0.01) and division (15% versus 6% of trials; χ^2 (1, 216) = 5.06, p = 0.02). The whole number part of the operands seemed to facilitate computational estimation on multiplication and division problems by allowing answers based solely on multiplying or dividing the whole number components.

Relations of explanations to direction of effect judgments. Type of explanation was strongly associated with accuracy of direction of effect judgments on multiplication and division problems with decimals below one (Table 3). This relation was only meaningful on these two types of problems, because accuracy was near ceiling for direction of effect judgments on problems with all other combinations of operation and operand size.

- - Insert Table 3 about here - -

As shown in Table 3, operation-and-operand explanations were associated with high accuracy on both multiplication and division problems with decimals below one. Despite this type of explanation being stated on only 26% of multiplication and 16% of division trials with operands below one, it was advanced on 65% of trials with correct multiplication judgments and 54% of trials with correct division judgments. These explanations probably reflect students grappling with how to integrate what they know about multiplication and division in general with what they know about results of those operations with numbers from 0-1.

In contrast, unconditional operation explanations were associated with very low accuracy on both multiplication and division problems with operands below one, less than 10% correct. In the context of these problems, citing the operation but not the operands, probably reflected the assumption that operand size is irrelevant to the direction of effect, as it is in adding and subtracting positive numbers and in multiplying and dividing numbers above one.

On these multiplication and division problems with decimals below one, explanations based on computational estimation were associated with low accuracy, though not as low as with unconditional operation explanations. One reason for the relatively low accuracy was that the two and three digit decimals in the problems made computational estimation difficult unless children rounded the decimals appropriately, which many did not. Another reason was that even when children were correct on the arithmetic, they often transformed answers they obtained so that they were consistent with their general assumption that multiplication yields answers larger than the operands, and division yields answers smaller than the number being divided. One child's explanations for the problems 0.87 * 0.291 and 0.87 ÷ 0.291 illustrates these difficulties. On the multiplication problem, the child said, "If you multiply 0.87 and 0.291, your answer comes to be around 2.793. 2.793 > 0.87." On the division problem, the child explained: "If you divide 87 by 29 you end up with 3 leaving you with 0.31 < 0.87."

Repeated addition explanations. Contrary to our expectation, none of the students' explanations referred to solving WD multiplication judgment problems with a decimal below one by using repeated addition -- estimating the result of adding the decimal the number of times indicated by the whole number (e.g., 5 * 0.291 interpreted as five iterations of 0.291). In contrast, many explanations were compatible with an unanticipated type of part-whole logic, in which the whole number in the WD problem is the whole and the decimal indicates multiplication by a number that is part of the unit "one" (e.g., "You are multiplying five by a number less than one so the solution is going to be less than one whole five"; "You are multiplying a number by a decimal, and that will make the number go down"; "You're losing stuff when you multiply by a decimal".) These were classified as "operation-and-operand" explanations in the overall categorization of explanations, but this subset of the category seemed worth separate consideration.

Consistent with these examples, on literally all WD multiplication problems with a decimal below one in which an explanation associated with a correct judgment treated the two operands asymmetrically, the decimal was treated as the operator and the whole number as the object of the operation. This approach was observed on 26% of WD multiplication problems. (Inter-rater agreement in coding this type of part-whole explanation was 93% (Cohen's kappa was = 0.77).

Magnitude comparison. Students correctly answered 88% of the decimal magnitude comparisons. Most students (76%) were accurate on more than 95% of the decimal comparisons; 9% of students consistently ignored the decimal points in the numbers being compared.

Number of correct decimal magnitude comparison and direction of effect judgments were weakly related. On the judgments only task, the relation was significant ($r = 0.35$, $p < 0.05$); on the judgments plus explanations condition, it was not ($r = 0.26$, n.s.).

General Discussion

This study extended prior ones in examining direction of effect judgments with decimals rather than fractions, problems involving a whole number and a rational number as well as two rational numbers, and measures that included confidence ratings and explanations of direction of effect judgments. Each of these features clarified the meaning of direction of effect judgments, sometimes in ways that differed from our expectations, and suggested means for improving instruction to increase students' understanding of rational number arithmetic.

One clear finding was that inaccurate direction of effect judgments for multiplication and division of fractions are not attributable only to difficulty understanding fraction notation. Identical difficulties were present with decimals, a notation that maps more transparently onto whole number notation. Thus, lack of understanding of the direction of effect of multiplying and dividing numbers below one is general to positive rational numbers, rather than being specific to fractions. Minimal correlations between accuracy of direction of effect judgments and accuracy on both magnitude comparison and arithmetic computation added evidence that this lack of understanding could not be attributed to lack of either magnitude or arithmetic knowledge.

Confidence ratings indicated differences between two groups of children. The half of children whose direction of effect judgments for decimal arithmetic invariably matched the pattern for the corresponding whole number arithmetic operation were highly confident in their incorrect judgments regarding multiplication and division of decimals below one. Their confidence in these incorrect judgments was not only very high in absolute terms, it was as high as their confidence in their correct judgments of the direction of effect of addition, subtraction, and multiplication of operands above one. Thus, these children's performance matched the strong conviction interpretation of direction of effect judgments.

In contrast, the half of children whose judgments less consistently matched the whole number pattern were less confident in some of their judgments. This was particularly the case for the older children (eighth graders) who were less confident in their direction of effect judgments involving decimals below one, especially on multiplication and division problems. This was consistent with the cognitive conflict interpretation. This finding might reflect the eighth graders whose judgments were less consistent beginning to suspect that the direction of effect of multiplication and division with numbers from zero to one differs from that with operands above one, but remaining uncertain. Examination of high school students' fraction and decimal direction of effect judgments and their confidence in those judgments could indicate whether understanding, or at least uncertainty, continues to grow with further mathematical experience.

The explanations data revealed a new phenomenon and improved understanding of another. The new phenomenon was that for both multiplication and division of decimals below one, direction of effect judgments vary greatly with the type of explanation that children generate. Explanations that noted both the arithmetic operation and whether the operands were above or below one were strongly associated with correct judgments; 90% of judgments that preceded such explanations were accurate. In contrast, less than 50% of judgments were correct when explanations cited only the type of operation, indicated reliance on computational estimation, or did not indicate any basis for the judgment. These data are consistent with the view that encoding not only the type of operation but also whether the operands are above or below one is essential to understanding rational number arithmetic.

The explanations data also changed our understanding of the finding that students were more accurate when judging the direction of effect on multiplication problems that involve a whole number and a decimal below one than when making such judgments on multiplication problems with two decimals below one. This effect was quite consistent; judgments were more accurate on all multiplication problems that included a whole number and a decimal below one than on any problem that included two decimals below one.

Although we predicted this finding, the explanations data revealed that our prediction was right for a wrong reason. The explanations showed no evidence for the hypothesized reliance on the logic of repeated addition to solve multiplication problems that involved a whole number and a decimal below one. Instead, most explanations that accompanied correct direction of effect judgments on such problems relied on a kind of part-whole logic. That is, the explanations emphasized that multiplying a whole number by a decimal less than one meant taking only a part of the whole number. In other words, rather than viewing the whole number as indicating the number of iterations of the decimal, children viewed the whole number as a whole and reasoned that multiplying by a number less than one would leave only part of the whole.

The same logic could have been applied to multiplication of two decimals between zero and one – there too, multiplying by a number less than one would leave only part of the original number – but it rarely was. One possibility is that greater familiarity with whole numbers might facilitate thinking about the effects of multiplying them by other numbers, perhaps through whole numbers being easier to encode as objects on which other multiplicands might operate. Another, non-exclusive, possibility is that the coincidence between the term "whole number" and that number serving as the whole in this context, promoted this reasoning.

The present research extended previous findings about direction of effect knowledge of decimals in at least three ways. One was demonstrating that similar findings emerge with more focused measures of direction of effect knowledge, judgments of the direction of effect in inequalities, as with the less focused measures of this knowledge used previously (selection of operations in word problems and unsolicited expressions of surprise) (Fischbein, Deri, Nello, & Marino, 1985; Graeber & Tirosh, 1990; Tirosh & Graeber, 1989). Another extension involved demonstrating that observations with fractions in these and our own previous study were not unique to fractions; rather, they extend to decimals as well. Third, the present findings narrowed the range of alternative explanations of the inaccurate judgments by showing that inaccurate direction of effect judgments were not due only to weak knowledge of operand magnitudes or computational procedures. Inaccurate direction of effect judgments with multiplication and division of decimals between 0 and 1 was observed even though most participants exhibited excellent understanding of decimal magnitudes and arithmetic procedures.

The findings also raise an intriguing theoretical question. Theories of error learning (e.g., Ohlsson, 1996; Ohlsson & Rees, 1991) propose that when people detect errors, they narrow their generalizations and subsequently err less often. The high frequency of direction of effect errors in the present study raises the issue of why such errors remain so frequent after years of fraction arithmetic experience. Do learners not notice the pattern that multiplying two numbers between zero and one always yields an answer smaller than either multiplicand? Do teachers not point out the pattern? Do children stop trying to make sense of rational number arithmetic, and therefore solely focus on executing procedures correctly rather than trying to identify relations between problems and answers? Specifying why these errors persist for so long, despite learners' substantial experience with rational number arithmetic, may prove useful in elaborating theories of error learning so that they can predict not only learning but also failures to learn.

Implications for Instruction

A general instructional implication of the present findings, especially taken together with the parallel findings of Siegler and Lortie-Forgues (2015) with fractions, is that at least some goals of the Common Core State Standards regarding understanding of rational number arithmetic are not yet being attained. For instance, interpreting multiplication as scaling (i.e., scaling up when multiplying by a number above one and scaling down when multiplying by a number below one) is one of the main learning goals of the Common Core (CCSSI, 2015) for fifth graders. If students had such understanding, they would have been much more accurate on the direction of effect task with both decimals and fractions than they turned out to be. To the extent that these findings are general, they suggest that current approaches to teaching conceptual understanding of rational number arithmetic need to be improved.

A more specific instructional implication was suggested by our finding that children were more accurate when multiplying a whole number by a decimal between zero and one than when multiplying two decimals of that size. This finding suggests that focusing on the former type of problem provides a useful transition between whole number multiplication and multiplication of two rational numbers. The fact that the part-whole logic was seen less often on multiplication problems with two decimals below one, despite being equally applicable to both types of problems, suggests that substantial transfer of the reasoning to such problems requires specific efforts to promote it. The instructional implication is that learning would benefit from teachers and textbooks presenting well-chosen analogies that highlight that the same reasoning applies to DD as to WD problems. Instruction based on structurally sound analogies has often proved effective in improving numerical understanding (e.g., Chen, Lu, & Holyoak, 2014; Opfer & Siegler, 2007; Sullivan & Barner, 2014). The clear parallels between multiplication of a whole number and a rational number between zero and one, and two rational numbers between zero and one, suggest that promoting analogies from the easier to the harder case could improve learning.

Another implication is that instruction should explicitly challenge students' belief that arithmetic with all numbers consistently works like arithmetic with whole numbers. Children whose direction of effect judgments invariably followed the whole number pattern were highly confident in the correctness of incorrect as well as correct judgments. Confidence is often a good thing, but misplaced confidence is not. One way to challenge the mistaken belief would be to focus students' attention on contradictory evidence. Students could predict the direction of effect of multiplication of rational numbers below one, and then compare their judgment with the actual answer generated by their own computation. Teachers could complement this activity with questions about why answers were wrong, as apparent contradictions alone could be ignored or attributed to calculation errors (Vosniadou, Ioannides, Dimitrakopoulou, Papademetriou, 2001). Confronting students with contradictory evidence is a common and effective teaching practice in other domains where misconceptions are frequent, such as science education (e.g., Chinn & Brewer, 1993). Moreover, people with high confidence in their errors have been found to be particularly responsive to feedback contradicting their beliefs (e.g., Butterfield & Metcalfe, 2001).

A further instructional implication is that students should be encouraged to consider both the size of the operands and the arithmetic operation when judging direction of effect of arithmetic operations. Explanations that cited both variables consistently accompanied correct judgments. By contrast, explanations that only cited the type of operation almost always accompanied incorrect judgments. Juxtaposing problems that involve operands below one with problems that involve operands above one, and asking students to reflect about why they need to consider the size of the operand as well as the operation, might prove effective at raising students' awareness of the relevance of both the operation and the operands to direction of effect judgments for multiplication. It might also help to increase their understanding of multiplication more generally.

Limitations and Future Directions

The present study has several limitations, each of which suggests directions for future research. One limitation is that our study does not address the effects of variations in mathematics curricula. Students who received more conceptually oriented instruction might show greater understanding of the direction of effect of rational number arithmetic operations. In a similar vein, the more accurate judgments on WD problems than on DD problems in Study 2 might reflect children encountering WD problems more often; without detailed knowledge of the input that children received, it was impossible to evaluate this interpretation, but the effects of curricula and instructional input more generally should be evaluated in future research.

Another limitation is that the present study did not directly compare direction of effect knowledge with decimals and fractions, and thus did not address the possibility that the notation moderates the strength of the observed effects. Future studies could test this possibility by presenting both fraction and decimal direction of effect problems to the same participants.

The present research also could not specify the role of teacher and textbook input on students' direction of effect knowledge. We attempted to contact the two teachers who taught the children in the study. One teacher indicated that she did not use a textbook but rather a variety of materials gathered from the internet; we could not locate the other teacher, who had left the school by the time we attempted to address this issue. The superior performance on WD relative to DD multiplication problems with operands between 0 and 1 might have been due to students encountering more WD than DD problems, or it might have been due to WD problems more often being presented with aids to conceptual understanding, such as manipulatives or number lines. In the absence of detailed data on the input that students received, this hypothesis could not be tested in the present study.

A further limitation of the present study is that idiosyncratic features of the task might have influenced students' reasoning. For instance, to allow identical operand orders for all four arithmetic operations without requiring understanding of negatives, we always presented the larger operand first and used it as the comparison answer (e.g., 5 * .291 > 5). This ordering, and the consequence of always having the whole number as the first operand on WD problems, might have influenced students' reasoning. In particular, presenting problems in which the whole number operand was second, such as 0.291 * 5 > 0.291, might have focused students' attention on the changes to 0.291 caused by being multiplied by 5 and thus led them to see the problem in terms of repeated addition. Another possibility is that phrasing the questions differently (e.g., "If you calculate how much 5 of the 0.291's is, will the answer be greater than 5?") might have revealed greater use of the repeated addition approach than the format used here (e.g., "Is 5 * 0.291 > 5?"). Testing the effects of these and other features of the direction of effect procedure would be valuable for evaluating the generality of the conclusions yielded by this study, as well as for suggesting ways of improving children's conceptual understanding of rational number arithmetic.

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Table 1

Percent Correct Direction of Effect Judgments for Decimal and Fraction Arithmetic by Operand Size and Arithmetic Operation.

Operand Sizes	Operation	Decimals	Fractions
Above one	Addition	88	92
	Subtraction	90	94
	Multiplication	84	92
	Division	89	70
Below one	Addition	87	89
	Subtraction	89	92
	Multiplication	20	31
	Division	19	47

Note: Percentages for fraction arithmetic in the right hand column are from grade peers in Siegler & Lortie-Forgues, 2015.

Table 2

Percent Correct Judgments on the Direction of Effect Judgments Task and on the Judgments Plus Explanations Task.

		Judgments Task		Judgments Plus Explanations Task	
Operand Size	Operation	DD Problems	WD Problems	DD Problems	WD Problems
Above one	Addition	97	96	94	96
	Multiplication	92	92	98	96
	Division	94	95	96	100
Below one	Addition	96	94	94	91
	Multiplication	31	47	33	43
	Division	21	24	24	28

Note: DD problems have two decimal operands; WD problems have one whole number and one decimal as operands.

Table 3

Percent Correct Direction of Effect Judgments on Multiplication and Division Items with Decimals Below One Associated with Each Explanation of Reasoning.

Type of explanation	Multiplication of decimals below 1	Division of decimals below 1
Operation and Operand	93	88
Unconditional Operation	5	2
Computational Estimation	44	38
Unspecified	44	45